Algebraic Characterizations of Regularity Properties in Bipartite Graphs

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Abstract

Regular and distance-regular characterizations of general graphs are well-known. In particular, the spectral excess theorem states that a connected graph Γ is distance-regular if and only if its spectral excess (a number that can be computed from the spectrum) equals the average excess (the mean of the numbers of vertices at extremal distance from every vertex). The aim of this paper is to derive new characterizations of regularity and distance-regularity for the more restricted family of bipartite graphs. In this case, some characterizations of (bi)regular bipartite graphs are given in terms of the mean degrees in every partite set and the Hoffman polynomial. Moreover, it is shown that the conditions for having distance-regularity in such graphs can be relaxed when compared with general graphs. Finally, a new version of the spectral excess theorem for bipartite graphs is presented.

1 Introduction

Bipartite graphs are combinatorial objects bearing some interesting symmetries. Thus, their spectra—eigenvalues of their adjacency matrices—are symmetric about zero, as the

corresponding eigenvectors come into pairs. Moreover, vertices in the same (respectively, different) independent set are always at even (respectively, odd) distance. Both characteristics have well-known consequences in most properties and parameters of such graphs. Roughly speaking, we could say that the conditions for a given property to be satisfied in a general graph can be somehow relaxed to guarantee the same property for a bipartite graph. The goal of this paper is to derive some new results supporting this claim in the framework of regular and distance-regular graphs, for which several characterizations of combinatorial or algebraic nature are known. Thus, the presented characterizations of bipartite distance-regular graphs involve parameters as the numbers of walks between vertices (entries of the powers of the adjacency matrix $\boldsymbol{A}$), the crossed local multiplicities (entries of the idempotents $\boldsymbol{E}_i$ or eigenprojectors), the predistance polynomials, etc. For instance, it is known that a regular graph Γ, with eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$ and diameter $D = d$, is distance-regular if and only if its idempotents $\boldsymbol{E}_1$ and $\boldsymbol{E}_d$ belong to the vector space $\mathcal{D}$ spanned by its distance matrices $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots \boldsymbol{A}_d$. In contrast with this, the same result holds in the case of bipartite graphs, but now only $\boldsymbol{E}_1 \in \mathcal{D}$ need to be required. Also, we derive a new version of the spectral excess theorem which gives a quasi-spectral characterization of distance-regularity for (regular) bipartite graphs.

We assume that the reader is familiar with the basic concepts on algebraic graph theory and, in particular, on distance-regular graphs. See, for instance, Cvetković, Doob, and Sachs [7], Biggs [2], Brouwer, Cohen, and Neumaier [4], Brouwer and Haemers [5], Van Dam, Koolen, and Tanaka [13], Fiol [16], and Godsil [21].

2 Preliminaries

Let $\Gamma = (V, A)$ be a (simple and connected) graph on $n = |V|$ vertices, with adjacency matrix $\boldsymbol{A}$, eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and spectrum

$$\operatorname{sp} \Gamma = \operatorname{sp} \boldsymbol{A} = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}, \tag{1}$$

where the different eigenvalues of Γ are in decreasing order, $\theta_0 > \theta_1 > \cdots > \theta_d$, and the superscripts stand for their multiplicities $m_i = m(\theta_i)$. In particular, note that when Γ is δ-regular, the largest eigenvalue is $\theta_0 = \lambda_1 = \delta$ and has multiplicity $m_0 = 1$ (as Γ is connected).

Recall also that Γ is bipartite if and only if it does not contain odd cycles. Then, its adjacency matrix is of the form

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix}.$$

(Here and henceforth it is assumed that the block matrices have the appropriate dimensions.) Moreover, for any polynomial $p \in \mathbb{R}_d[x]$ with even and odd parts p_0 and p_1, we have

$$p(\boldsymbol{A}) = p_0(\boldsymbol{A}) + p_1(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D} \end{pmatrix} + \begin{pmatrix} \boldsymbol{O} & \boldsymbol{M} \\ \boldsymbol{M}^\top & \boldsymbol{O} \end{pmatrix}. \tag{2}$$

Also, the spectrum of Γ is symmetric about zero: $\theta_i = -\theta_{d-i}$ and $m_i = m_{d-i}$, $i = 0, 1, \ldots, d$. (In fact, a well-known result states that a connected graph Γ is bipartite if and only if $\theta_0 = -\theta_d$; see, for instance, Cvetković, Doob and Sachs [7].) This is due to the fact that, if $(\boldsymbol{u}|\boldsymbol{v})^\top$ is a (right) eigenvector with eigenvalue θ_i, then $(\boldsymbol{u}| -\boldsymbol{v})^\top$ is an eigenvector for the eigenvalue $-\theta_i$. As shown below, a similar symmetry also applies to the entries of the (*principal*) *idempotents* $\boldsymbol{E}_i$ representing the projections onto the eigenspaces $\mathcal{E}_i = \ker(\boldsymbol{A} - \theta_i\boldsymbol{I})$, $i = 0, 1, \ldots, d$. To see this, first recall that, for any graph with eigenvalue θ_i having multiplicity m_i, its corresponding idempotent can be computed as $\boldsymbol{E}_i = \boldsymbol{U}_i\boldsymbol{U}_i^\top$, where $\boldsymbol{U}_i$ is the $n \times m_i$ matrix whose columns form an orthonormal basis of $\mathcal{E}_i$. For instance, when Γ is a δ-regular graph on n vertices its largest eigenvalue $\theta_0 = \delta$ has eigenvector $\boldsymbol{j}$, the all-1 (column) vector, and corresponding idempotent $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{j}\boldsymbol{j}^\top = \frac{1}{n}\boldsymbol{J}$, where $\boldsymbol{J}$ is the all-1 matrix. Alternatively, we can also compute the idempotents as $\boldsymbol{E}_i = L_i(\boldsymbol{A})$ where L_i is the Lagrange interpolating polynomial of degree d satisfying $L_i(\theta_i) = 1$ and $L_i(\theta_j) = 0$ for $j \neq i$. The entries of the idempotents $m_{uv}(\theta_i) = (\boldsymbol{E}_i)_{uv}$ are called *crossed uv-local multiplicities* and, by taking $p = x^\ell$, $\ell \geq 0$, they allow us to compute the number of ℓ-walks between any two vertices:

$$a_{uv}^{(\ell)} = (\boldsymbol{A}^\ell)_{uv} = \sum_{i=0}^{d} m_{uv}(\theta_i)\theta_i^\ell,$$

(see Fiol, Garriga and Yebra [20], Dalfó, Fiol and Garriga [10] and Dalfó, van Dam, Fiol, Garriga and Gorissen [8]). In particular, when $u = v$, $m_u(\theta_i) = m_{uu}(\theta_i)$ are the so-called *local multiplicities* of vertex u, satisfying $\sum_{i=0}^{d} m_u(\theta_i) = 1$ for $u \in V$, and $\sum_{u \in V} m_u(\theta_i) = m_i$ for $i = 0, 1, \ldots, d$ (see Fiol and Garriga [17]).

Let $\mathrm{dist}(u, v)$ denote the distance between vertices u, v. From any of the above expressions for $\boldsymbol{E}_i$, we infer that, when Γ is bipartite, the crossed local multiplicities satisfy:

- $m_{uv}(\theta_i) = m_{uv}(\theta_{d-i})$, $i = 0, 1, \ldots, d$, if $\mathrm{dist}(u, v)$ is even.

- $m_{uv}(\theta_i) = -m_{uv}(\theta_{d-i})$, $i = 0, 1, \ldots, d$, if $\mathrm{dist}(u, v)$ is odd.

In particular, as $\mathrm{dist}(u, u) = 0$, the local multiplicities bear the same symmetry as the standard multiplicities: $m_u(\theta_i) = m_u(\theta_{d-i})$ for any vertex $u \in V$ and eigenvalue θ_i, $i = 0, 1, \ldots, d$.

From the above results, notice that, when Γ is regular and bipartite, we have $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J}$ (as mentioned before) and

$$\boldsymbol{E}_d = \frac{1}{n}\begin{pmatrix} \boldsymbol{J} & -\boldsymbol{J} \\ -\boldsymbol{J} & \boldsymbol{J} \end{pmatrix}. \tag{3}$$

As proved by Haemers [24], the following result is a direct consequence of the *Interlacing Theorem*.

Theorem 2.1 ([24]) *Let $\boldsymbol{A}$ be a symmetric matrix partitioned into m^2 blocks $\boldsymbol{A}_{ij}$, $i, j = 1, 2, \ldots, m$, with $m < n$, such that $\boldsymbol{A}_{ii}$ is a square matrix for any $i = 1, 2, \ldots, m$. Let $\boldsymbol{B} = (b_{ij})$ be the $m \times m$ matrix with b_{ij} being the average row sum of $\boldsymbol{A}_{ij}$, for $i, j = 1, 2, \ldots, m$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Then,*

(a) *The eigenvalues of $\boldsymbol{B}$ interlace the eigenvalues of $\boldsymbol{A}$. That is,*

$$\lambda_{n-m+i} \leq \mu_i \leq \lambda_i, \quad i = 1, 2, \ldots, m.$$

(b) *If the interlacing is tight, that is, for some $0 \leq k \leq m$, $\mu_i = \lambda_i$ for $i = 1, 2, \ldots, k$ and $\mu_i = \lambda_{n-m+i}$ for $i = k+1, k+2, \ldots, m$, then $\boldsymbol{A}_{ij}$ has constant row and column sums for $i, j = 1, 2, \ldots, m$.*

3 Regular graphs

We begin our study with a simple result characterizing biregularity in a bipartite graph.

3.1 Spectrum and regularity

For a given graph $\Gamma = (V, E)$, let δ_u denote the degree of vertex $u \in V$. Then, the *average degree* of Γ is defined by

$$\overline{\delta} = \frac{1}{n} \sum_{u \in V} \delta_u = \frac{1}{n} \operatorname{tr} \boldsymbol{A}^2 = \frac{1}{n} \sum_{i=1}^{n} \lambda_i^2. \tag{4}$$

Thus, if in Theorem 2.1 we consider the trivial partition with $m = 1$, the *average quotient matrix* $\boldsymbol{B}$ (whose entries are the average row sums of the blocks of $\boldsymbol{A}$) has eigenvalue $\mu_1 = \overline{\delta}$ and, hence,

$$\overline{\delta} \leq \lambda_1, \tag{5}$$

with equality if and only if Γ is $\overline{\delta}$-regular. (Combining this with (4), we see that regularity is a property that can be deduced from the spectrum.)

Let us show that there is an analogous result for bipartite graphs. A bipartite graph $\Gamma = (V_1 \cup V_2, E)$ is called (δ_1, δ_2)-*biregular* when all n_1 vertices of V_1 have degree δ_1, and the n_2 vertices of V_2 have degree δ_2. So, by counting in two different ways the number of edges $m = |E|$, we have that $n_1 \delta_1 = n_2 \delta_2$. Also, it is well known that, for such a graph, $\theta_0 = \sqrt{\delta_1 \delta_2}$ (see, for instance, Godsil and Royle [23, pp. 172–173]). For a general bipartite graph with independent sets V_1, V_2, define $\overline{\delta}_1$ and $\overline{\delta}_2$ as the average degree of the vertices of V_1 and V_2, that is, $\overline{\delta}_1 = \frac{1}{n_1} \sum_{u \in V_1} \delta_u$ and $\overline{\delta}_2 = \frac{1}{n_2} \sum_{u \in V_2} \delta_u$.

Proposition 3.1 *Let $\Gamma = (V_1 \cup V_2, E)$ be a bipartite graph with $n = n_1 + n_2$ vertices, average degrees $\overline{\delta}_1$ and $\overline{\delta}_2$ and maximum eigenvalue λ_1. Then,*

$$\sqrt{\overline{\delta}_1 \overline{\delta}_2} \leq \lambda_1 \tag{6}$$

and equality holds if and only if Γ is $(\overline{\delta}_1, \overline{\delta}_2)$-biregular.

Proof. Let

$$A = \begin{pmatrix} \mathbf{0} & A_{1,2} \\ A_{2,1} & \mathbf{0} \end{pmatrix}$$

be the adjacency matrix of Γ and consider the natural average quotient matrix with $m = 2$

$$B = \begin{pmatrix} 0 & \overline{\delta}_1 \\ \overline{\delta}_2 & 0 \end{pmatrix}.$$

Then, since its eigenvalues are $\pm\sqrt{\overline{\delta}_1 \overline{\delta}_2}$, Theorem 2.1(a) gives

$$\mu_1 = \sqrt{\overline{\delta}_1 \overline{\delta}_2} \leq \lambda_1.$$

Moreover, in case of equality, $\mu_2 = -\mu_1 = -\lambda_1 = \lambda_n$, so that the interlacing is tight and Theorem 2.1(b) implies the biregularity of Γ. $\square$

In fact, we can derive (5) and (6) and other similar results from the well-known result from linear algebra known as the *Rayleigh's principle* or *Rayleigh's inequalities* (see, for instance, Godsil and Royle [23, p. 202]). This result states that, if $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are the eigenvectors corresponding to $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ respectively, and for some $1 \leq i \leq j \leq n$ we have $\boldsymbol{u} \in \text{span}\{\boldsymbol{u}_i, \boldsymbol{u}_{i+1}, \ldots, \boldsymbol{u}_j\}$, then

$$\lambda_j \leq \frac{\langle \boldsymbol{u}, A\boldsymbol{u}\rangle}{||\boldsymbol{u}||^2} \leq \lambda_i.$$

Moreover, equality on the left (respectively, right) implies that $\boldsymbol{u}$ is a λ_j-eigenvector (respectively, λ_i-eigenvector) of A. Then, the clue is to make the "right choice" of $\boldsymbol{u}$. For instance, let us consider the derivation of the two previous results:

1. If $\boldsymbol{u} = \boldsymbol{j}$, we have $\langle \boldsymbol{j}, A\boldsymbol{j}\rangle = \sum_{u \in V} \delta_u$, $||\boldsymbol{j}||^2 = n$, and we get (5).

2. Assume that Γ is bipartite with stable sets V_1, V_2, numbers of vertices $n_1 = |V_1|$, $n_2 = |V_2|$, and average degrees $\overline{\delta}_1$, $\overline{\delta}_2$. Notice that $n_1\overline{\delta}_1 + n_2\overline{\delta}_2 = 2m$. Then, if $\boldsymbol{u} = (\sqrt{\overline{\delta}_1}\boldsymbol{j} | \sqrt{\overline{\delta}_2}\boldsymbol{j})$ with $||\boldsymbol{u}||^2 = 2m$, we get

$$\frac{\langle \boldsymbol{u}, A\boldsymbol{u}\rangle}{||\boldsymbol{u}||^2} = \sqrt{\overline{\delta}_1 \overline{\delta}_2} \leq \lambda_1$$

and, from the above comments, equality is attained when Γ is biregular.

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3.2 Polynomials and regularity

The *predistance polynomials* $p_0, p_1, \ldots, p_d$, with degree $\deg p_i = i$, $i = 0, 1, \ldots, d$, associated to a given graph Γ with spectrum $\operatorname{sp}\Gamma$ as in (1), are a sequence of orthogonal polynomials with respect to the scalar product

$$\langle f, g \rangle_\Gamma = \frac{1}{n} \operatorname{tr}(f(\boldsymbol{A})g(\boldsymbol{A})) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\theta_i) g(\theta_i),$$

normalized in such a way that $\|p_i\|_\Gamma^2 = p_i(\theta_0)$ (this makes sense since, as it is well-known, we always have $p_i(\theta_0) > 0$; for a simple explanation see, for instance, Van Dam [12]). As every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$x p_i = \beta_{i-1} p_{i-1} + \alpha_i p_i + \gamma_{i+1} p_{i+1}, \qquad i = 0, 1, \ldots, d, \tag{7}$$

where $\beta_{-1} = \gamma_{d+1} = 0$ and the other constants β_i, γ_i are nonzero, initialized with $p_0 = 1$ and $p_1 = x$. Moreover, if Γ is bipartite, the symmetry of such a scalar product yields that $\alpha_i = 0$ for every $i = 1, 2, \ldots, d$, and p_i is even (respectively, odd) for even (respectively, odd) degree i. In terms of the predistance polynomials, the *preHoffman polynomial* is $H = p_0 + p_1 + \cdots + p_d$, and satisfies $H(\theta_0) = n$ (the order of the graph) and $H(\theta_i) = 0$ for $i = 1, 2, \ldots, d$ (for more details about the predistance polynomials, see Cámara, Fàbrega, Fiol and Garriga [6]). In [25], Hoffman proved that a (connected) graph Γ is regular if and only if $H(\boldsymbol{A}) = \boldsymbol{J}$, in which case H becomes the *Hoffman polynomial*. (In fact, H is the unique polynomial of degree at most d satisfying this property.) Furthermore, when Γ is regular and bipartite, the even and odd parts of H, $H_0 = \sum_{i \text{ even}} p_i$ and $H_1 = \sum_{i \text{ odd}} p_i$, satisfy, by (2):

$$H_0(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J} \end{pmatrix} \quad \text{and} \quad H_1(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix}. \tag{8}$$

As far as we know, the following theorem is new and can be seen as the biregular counterpart of Hoffman's result.

Theorem 3.2 *A bipartite graph $\Gamma = (V_1 \cup V_2, E)$ with $n = |V_1| + |V_2| = n_1 + n_2$ vertices is (δ_1, δ_2)-biregular if and only if the odd part of its preHoffman polynomial satisfies*

$$H_1(\boldsymbol{A}) = \alpha \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix} \tag{9}$$

with $\alpha = \dfrac{n_1 + n_2}{2\sqrt{n_1 n_2}} = \dfrac{\delta_1 + \delta_2}{2\sqrt{\delta_1 \delta_2}}$.

Proof. Assume first that Γ is biregular with degrees, say, δ_1 and δ_2. Then, $\theta_0 = -\theta_d = \sqrt{\delta_1 \delta_2}$ with respective (column) eigenvectors $\boldsymbol{u} = (\sqrt{\delta_1}\boldsymbol{j} | \sqrt{\delta_2}\boldsymbol{j})^\top$ and $\boldsymbol{v} =$

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$(\sqrt{\delta_1}\boldsymbol{j}| - \sqrt{\delta_2}\boldsymbol{j})^\top$, with $\boldsymbol{j}$'s being all-1 (row) vectors with the appropriate lengths. Therefore, the respective idempotents are

$$
\begin{aligned}
\boldsymbol{E}_0 &= \frac{1}{\|\boldsymbol{u}\|^2}\boldsymbol{u}\boldsymbol{u}^\top = \frac{1}{n_1\delta_1 + n_2\delta_2}\begin{pmatrix} \delta_1\boldsymbol{J} & \sqrt{\delta_1\delta_2}\boldsymbol{J} \\ \sqrt{\delta_1\delta_2}\boldsymbol{J} & \delta_2\boldsymbol{J} \end{pmatrix}, \\
\boldsymbol{E}_d &= \frac{1}{\|\boldsymbol{v}\|^2}\boldsymbol{v}\boldsymbol{v}^\top = \frac{1}{n_1\delta_1 + n_2\delta_2}\begin{pmatrix} \delta_1\boldsymbol{J} & -\sqrt{\delta_1\delta_2}\boldsymbol{J} \\ -\sqrt{\delta_1\delta_2}\boldsymbol{J} & \delta_2\boldsymbol{J} \end{pmatrix}.
\end{aligned}
$$

As $H_1(x) = \frac{1}{2}[H(x) - H(-x)]$ with $H(\theta_0) = n$ and $H(\theta_i) = 0$ for any $i \neq 0$, we have that $H_1(\theta_0) = -H_1(\theta_d) = n/2$ and $H_1(\theta_i) = 0$ for $i \neq 0, d$. Hence, using the properties and the above expressions of the idempotents,

$$
\begin{aligned}
H_1(\boldsymbol{A}) &= \sum_{i=0}^d H_1(\theta_i)\boldsymbol{E}_i = H_1(\theta_0)\boldsymbol{E}_0 + H_1(\theta_d)\boldsymbol{E}_d \\
&= \frac{n}{2}(\boldsymbol{E}_0 - \boldsymbol{E}_d) = \frac{n\sqrt{\delta_1\delta_2}}{n_1\delta_1 + n_2\delta_2}\begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix},
\end{aligned}
$$

and the result follows since $n_1\delta_1 = n_2\delta_2$.

Conversely, if (9) holds, and $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix}$, the equality $\boldsymbol{A}H_1(\boldsymbol{A}) = H_1(\boldsymbol{A})\boldsymbol{A}$ yields

$$
\begin{pmatrix} \boldsymbol{B}\boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B}^\top\boldsymbol{J} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}\boldsymbol{B}^\top & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J}\boldsymbol{B} \end{pmatrix}.
$$

Thus, $(\boldsymbol{B}\boldsymbol{J})_{uv} = (\boldsymbol{J}\boldsymbol{B}^\top)_{uv}$ implies that $\delta_u = \delta_v$ for any two vertices $u, v \in V_1$, whereas $(\boldsymbol{B}^\top\boldsymbol{J})_{wz} = (\boldsymbol{J}\boldsymbol{B})_{wz}$ means that $\delta_w = \delta_z$ for any two vertices $w, z \in V_2$. Thus, Γ is biregular and the proof is complete. $\square$

Notice that the constant α is the ratio between the arithmetic and geometric means of the numbers n_1, n_2. Hence, (9) holds with $\alpha = 1$ if and only if $n_1 = n_2$ or, equivalently, when Γ is regular.

In fact, the above result can be reformulated in the following way:

Theorem 3.3 *A bipartite graph Γ, with $n = n_1 + n_2$ vertices and distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$, is connected and biregular if and only if there exists a polynomial P satisfying*

$$
P(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix}. \tag{10}
$$

Moreover, $P = \frac{2\sqrt{n_1 n_2}}{n_1 + n_2}H_1$, where H_1 is the odd part of the preHoffmann polynomial of Γ, so that $P(\theta_0) = -P(\theta_d) = \sqrt{n_1 n_2}$, and $P(\theta_i) = 0$ for $i \neq 0, d$.

$\boldsymbol{Proof.}$ We only need to prove sufficiency, since necessity has already been proved in Theorem 3.2. If (10) holds, we have that for any two vertices u, v in different partite sets,

$(P(\boldsymbol{A}))_{uv} = 1 \neq 0$ implies that there is some u-v path. Also, if u, v are vertices in the same partite set, for any vertex w in the other partite set there exist u-w and v-w paths, thus assuring again the existence of some u-v path. Consequently, Γ is connected and, from the same reasoning as above, biregular with degrees, say, δ_1, δ_2. Now, let us prove that P is as claimed. Since $\boldsymbol{u} = (\sqrt{\delta_1}\boldsymbol{j}|\sqrt{\delta_2}\boldsymbol{j})^\top$ is an eigenvector of $\boldsymbol{A}$ for the eigenvalue $\theta_0 = \sqrt{\delta_1 \delta_2}$, we have

$$P(\boldsymbol{A})\boldsymbol{u} = P(\theta_0) \begin{pmatrix} \sqrt{\delta_1}\boldsymbol{j}^\top \\ \sqrt{\delta_2}\boldsymbol{j}^\top \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \sqrt{\delta_1}\boldsymbol{j}^\top \\ \sqrt{\delta_2}\boldsymbol{j}^\top \end{pmatrix} = \begin{pmatrix} n_2\sqrt{\delta_2}\boldsymbol{j}^\top \\ n_1\sqrt{\delta_1}\boldsymbol{j}^\top \end{pmatrix},$$

whence $P(\theta_0) = \frac{n_2\sqrt{\delta_2}}{\sqrt{\delta_1}} = \sqrt{n_1 n_2}$. Similarly, by using that $\boldsymbol{v} = (\sqrt{\delta_1}\boldsymbol{j}| - \sqrt{\delta_2}\boldsymbol{j})^\top$ is an eigenvector of $\boldsymbol{A}$ for the eigenvalue $\theta_d = -\sqrt{\delta_1 \delta_2}$, we get $P(\theta_d) = -\sqrt{n_1 n_2}$.

Finally, assume that $\boldsymbol{w} = (\boldsymbol{w}_1|\boldsymbol{w}_2)^\top$ is an eigenvector of $\boldsymbol{A}$ with eigenvalue θ_i, $i \neq 0, d$. Then, from the orthogonality conditions $\langle \boldsymbol{w}, \boldsymbol{u} \rangle = 0$ and $\langle \boldsymbol{w}, \boldsymbol{v} \rangle = 0$ we have, respectively, that $\sqrt{\delta_1}\langle \boldsymbol{w}_1, \boldsymbol{j} \rangle + \sqrt{\delta_2}\langle \boldsymbol{w}_2, \boldsymbol{j} \rangle$ and $\sqrt{\delta_1}\langle \boldsymbol{w}_1, \boldsymbol{j} \rangle - \sqrt{\delta_2}\langle \boldsymbol{w}_2, \boldsymbol{j} \rangle = 0$. Therefore, $\langle \boldsymbol{w}_1, \boldsymbol{j} \rangle = \langle \boldsymbol{w}_2, \boldsymbol{j} \rangle = 0$. Moreover,

$$P(\boldsymbol{A})\boldsymbol{w} = P(\theta_i) \begin{pmatrix} \boldsymbol{w}_1^\top \\ \boldsymbol{w}_2^\top \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{w}_1^\top \\ \boldsymbol{w}_2^\top \end{pmatrix} = \begin{pmatrix} n_2\langle \boldsymbol{w}_1, \boldsymbol{j} \rangle \boldsymbol{j}^\top \\ n_1\langle \boldsymbol{w}_2, \boldsymbol{j} \rangle \boldsymbol{j}^\top \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix},$$

and, since $\boldsymbol{w} \neq \boldsymbol{0}$, $P(\theta_i) = 0$, as claimed. $\square$

4 Distance-regular graphs

In this section we give some new characterizations of bipartite distance-regular graphs. As commented in the introduction, a general phenomenon is that the conditions for a bipartite graph to be distance-regular can be relaxed in comparison with the analogues for general graphs. To prove and compare our results, we first recall some basic facts from the theory of distance-regular graphs and some of their known characterizations.

4.1 Some characterizations of distance-regularity

Let Γ be a graph with diameter D, adjacency matrix $\boldsymbol{A}$, and $d + 1$ distinct eigenvalues. Let $\boldsymbol{A}_i$, $i = 0, 1, \ldots$, be the *distance-i matrix* of Γ, with entries $(\boldsymbol{A}_i)_{uv} = 1$ if $\text{dist}(u, v) = i$ and $(\boldsymbol{A}_i)_{uv} = 0$ otherwise (if $i > D$, we have $\boldsymbol{A}_i = \boldsymbol{O}$). Then,

$$\mathcal{A} = \mathbb{R}_d[\boldsymbol{A}] = \text{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\}$$

is an algebra, with the ordinary product of matrices and orthogonal basis $\{\boldsymbol{E}_0, \boldsymbol{E}_1, \ldots, \boldsymbol{E}_d\}$ and $\{p_0(\boldsymbol{A}), p_1(\boldsymbol{A}), \ldots, p_d(\boldsymbol{A})\}$, called the *adjacency algebra*; whereas

$$\mathcal{D} = \text{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_D\}$$

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forms an algebra with the entrywise or Hadamard product "∘" of matrices, defined by $(\boldsymbol{X} \circ \boldsymbol{Y})_{uv} = (\boldsymbol{X})_{uv}(\boldsymbol{Y})_{uv}$. We call $\mathcal{D}$ the *distance* ∘-*algebra*. Note that, when Γ is regular, $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{J} \in \mathcal{A} \cap \mathcal{D}$ since $\boldsymbol{J} = H(\boldsymbol{A}) = \sum_{i=0}^{D} \boldsymbol{A}_i$. Thus, $\dim(\mathcal{A} \cap \mathcal{D}) \geq 3$, if Γ is not a complete graph (in this exceptional case, $\boldsymbol{J} = \boldsymbol{I} + \boldsymbol{A}$). In this algebraic context, an important result is that Γ is distance-regular if and only if $\mathcal{A} = \mathcal{D}$, which is therefore equivalent to $\dim(\mathcal{A} \cap \mathcal{D}) = d + 1$ (and hence $d = D$); see, for instance, Biggs [2] or Brouwer, Cohen and Neumaier [4]. This leads to some characterizations of distance-regularity in terms of some constants p_{ji}, q_{ij} and $a_i^{(\ell)}$ (see also Rowlinson [26]): A graph Γ is distance-regular if and only if any of the following conditions holds.

(a) $\boldsymbol{A}_i \boldsymbol{E}_j = p_{ji} \boldsymbol{E}_j, \quad i, j = 0, 1, \ldots, d(= D)$.

(b) $\boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij} \boldsymbol{A}_i, \quad i, j = 0, 1, \ldots, d$.

(c) $\boldsymbol{A}^\ell \circ \boldsymbol{A}_i = a_i^{(\ell)} \boldsymbol{A}_i, \quad i, \ell = 0, 1, \ldots, d$.

When the conditions (a) hold, the predistance polynomials become the *distance polynomials* satisfying $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$, $i = 0, 1, 2, \ldots, d$ (see, for example, Bannai and Ito [1]). Moreover, for general graphs with $D \leq d$, the conditions (a) are a characterization of the so-called *distance-polynomial graphs*, introduced by Weichsel [27] (see also Beezer [3] and Dalfó, van Dam, Fiol, Garriga and Gorissen [8]). This is equivalent to $\mathcal{D} \subset \mathcal{A}$ (but not necessarily $\mathcal{D} = \mathcal{A}$); that is, every distance matrix $\boldsymbol{A}_i$ is a polynomial (not necessarily p_i) in $\boldsymbol{A}$. In contrast with this, the conditions (b) and (c) are equivalent to $\mathcal{A} \subset \mathcal{D}$ and hence to $\mathcal{A} = \mathcal{D}$ (which implies $d = D$) as $\dim \mathcal{A} \geq \dim \mathcal{D}$.

As summarized in the following theorem, the above conditions for distance-regularity can be relaxed if we assume some extra natural hypothesis (such as regularity) thus giving more 'economical' characterizations.

Theorem 4.1 ([19, 15, 8]) *A regular graph Γ with $d + 1$ distinct eigenvalues, diameter $D = d$, and idempotents $\boldsymbol{E}_1, \boldsymbol{E}_d$ is distance-regular if and only if any of the following conditions holds:*

(a) $\boldsymbol{A}_d \in \mathcal{A}, \quad or \quad \boldsymbol{A}_d = p_d(\boldsymbol{A}), \quad or \quad \boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = d - 2, d - 1$.

(b) $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 1, d$.

(c) $\boldsymbol{A}^\ell \circ \boldsymbol{A}_i \in \mathcal{D}$ *for* $\ell = i, i + 1 \leq d - 1$.

4.2   The case of bipartite graphs

In the following result we prove that for bipartite graphs the above conditions can be further relaxed giving new characterizations for bipartite distance-regular graphs.

Theorem 4.2 *A regular bipartite graph Γ with diameter $D = d$, idempotent $\boldsymbol{E}_1$, and predistance polynomials $p_0, p_1, \ldots, p_d$ is distance-regular if and only if*

(a) $\boldsymbol{A}_{d-2} = p_{d-2}(\boldsymbol{A})$.

(b) $\boldsymbol{E}_1 \in \mathcal{D}$.

(c) $a_{uv}^{(\ell)} = a_i^{(\ell)}$ *for $\ell = i \leq D - 2$.*

Proof. (a) From Theorem 4.1(a), it suffices to prove that $p_d(\boldsymbol{A}) = \boldsymbol{A}_d$. Let u, v two vertices at distance $\mathrm{dist}(u, v) = i$. Then, if $i = d$, $(p_d(\boldsymbol{A}))_{uv} = (H(\boldsymbol{A}))_{uv} = 1$. Moreover, if i and d have distinct parity, $(p_d(\boldsymbol{A}))_{uv} = 0$. Thus, we only need to consider the case when $i(< d)$ and d have the same parity. If $i = d - 2$, from the hypothesis we know that $(p_{d-2}(\boldsymbol{A}))_{uv} = 1$ and, by (8), $((p_d + p_{d-2} + \cdots)(\boldsymbol{A}))_{uv} = 1$, whence $(p_d(\boldsymbol{A})_{uv}) = 0$. Finally, if $i < d - 2$, the recurrence (7), yields

$$(\boldsymbol{A}p_{d-1}(\boldsymbol{A}))_{uv} = \beta_{d-2}(\boldsymbol{A}_{d-2})_{uv} + \gamma_d(p_d(\boldsymbol{A}))_{uv} = \gamma_d(p_d(\boldsymbol{A}))_{uv},$$

with first term

$$(\boldsymbol{A}p_{d-1}(\boldsymbol{A}))_{uv} = \sum_{w \in V} a_{uw}(p_{d-1}(\boldsymbol{A}))_{wv} = \sum_{w \in \Gamma(u)} (p_{d-1}(\boldsymbol{A}))_{wv} = 0,$$

since $\mathrm{dist}(v, w)$ and i have distinct parity. Consequently, as $\gamma_d \neq 0$, $(p_d(\boldsymbol{A}))_{uv} = 0$ and $p_d(\boldsymbol{A}) = \boldsymbol{A}_d$, as claimed. Characterization (b) is a consequence of Theorem 4.1(b) since, under the hypotheses, (3) yields

$$\boldsymbol{E}_d = \tfrac{1}{n} \left(\sum_{\substack{i = 0 \\ i \text{ even}}}^{d} \boldsymbol{A}_i - \sum_{\substack{i = 0 \\ i \text{ odd}}}^{d} \boldsymbol{A}_i \right) \in \mathcal{D}.$$

Finally, (c) follows from the fact that, since Γ is bipartite, there are no walks of length $\ell = i + 1$ between vertices at distance i and, thus, $a_i^{(i+1)} = 0$. Moreover, as Γ is δ-regular and $D = d$, we have $a_{d-1}^{(d-1)} = \frac{1}{\delta} a_d^{(d)} = \frac{\pi_0}{n\delta}$ because the number of d-walks between any two vertices u, v at distance d, is a constant: $a_{uv}^{(d)} = (\boldsymbol{A}^d)_{uv} = \frac{\pi_0}{n}[H(\boldsymbol{A})]_{uv} = \frac{\pi_0}{n} = a_d^{(d)}$. $\quad\square$

In [11] Van Dam proved that a connected regular graph with four distinct eigenvalues is *walk-regular* (that is, for every $\ell \geq 0$, the number of closed ℓ-walks rooted at a given vertex is a constant through the graph, see Godsil and McKay [22]). Also, Dalfó, van Dam, Fiol, Garriga and Gorissen [8] observed that the same is true for a every connected regular bipartite graph with five distinct eigenvalues. (For properties about bipartite biregular graphs with five eigenvalues, see Van Dam and Spence [14].) As a consequence of the above theorem, we also have the following:

Corollary 4.3   (a) *Every bipartite δ-regular graph Γ with $d + 1 = 4$ distinct eigenvalues and diameter $D = 3$ is distance-regular.*

(b) *Every bipartite δ-regular graph* Γ *with* $d+1 = 5$ *distinct eigenvalues, diameter* $D = 4$, *and with every pair of nonadjacent vertices having* c *common neighbors is distance-regular.*

Proof. Just notice that, in case (a), $p_{d-2} = p_1 = x$ is the distance-1 polynomial, whereas in (b), under the hypothesis, $p_{d-2} = p_2 = \frac{1}{c}(x^2 - \delta)$ is the distance-2 polynomial. Hence, in both cases, Theorem 4.2(a) applies. $\square$

4.3 The spectral excess theorem for bipartite graphs

Let Γ be a graph with $d+1$ eigenvalues and distance-d matrix $\boldsymbol{A}_d$. Then the *excess* of a vertex u is the number of vertices at distance d from u, that is $\mathrm{exc}(u) = |\Gamma_d(u)|$, and the *average excess* is $\overline{\delta}_d = \frac{1}{n} \sum_{u \in V} \mathrm{exc}(u)$. Moreover, the *spectral excess* is the value at λ_0 of the highest degree predistance polynomial and it can be computed by using $\mathrm{sp}\,\Gamma$ through the formula

$$p_d(\lambda_0) = n \left(\sum_{i=0}^{d} \frac{\pi_0^2}{\pi_i^2} \right)^{-1},$$

where $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$ for $i = 0, 1, \ldots, d$. By using these parameters, the spectral excess theorem, due to Fiol and Garriga [17], provides a quasi-spectral characterization of distance-regularity for regular connected graphs and reads as follows (see Van Dam [12] or Fiol, Gago and Garriga [18] for short proofs):

Theorem 4.4 ([17, 12, 18]) *A connected regular graph with* $d+1$ *distinct eigenvalues is distance-regular if and only if its average excess equals its spectral excess:*

$$\overline{\delta}_d = p_d(\lambda_0).$$

The proof of this theorem is based on the characterization of Theorem 4.1(a) and, hence, it is natural to devise an analogous result for bipartite graphs, which is given in terms of the following parameters. Given a graph Γ, let ω_i be the leading coefficient of its predistance polynomial p_i, and consider the average numbers of vertices at distance i and shortest i-paths; that is,

$$\overline{\delta}_i = \frac{1}{n} \sum_{u \in V} |\Gamma_i(u)| \quad \text{and} \quad \overline{a}_i^{(i)} = \frac{1}{n\overline{\delta}_i} \sum_{\mathrm{dist}(u,v)=i} a_{uv}^{(i)},$$

respectively. Then, we have the following result:

Theorem 4.5 *A regular bipartite graph* Γ *with* $d+1$ *distinct eigenvalues and diameter* $D = d$ *is distance-regular if and only if*

$$\overline{a}_{d-2}^{(d-2)} = 1/\omega_{d-2} \quad \text{and} \quad \overline{\delta}_{d-2} = p_{d-2}(\lambda_0).$$

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Proof. The necessity of both conditions follows easily by considering that, if Γ is distance-regular, then $p_{d-2}(\boldsymbol{A}) = \boldsymbol{A}_{d-2}$, whereas the sufficiency follows from [9, Prop. 2.4(a3)], assuring that $p_{d-2}(\boldsymbol{A}) = \boldsymbol{A}_{d-2}$, and Theorem 4.2($a$). $\quad\square$

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